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INVERSE PROBLEMS FOR THE STATIONARY TRANSPORT EQUATION IN THE DIFFUSION SCALING*

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Abstract. We consider the inverse problem of reconstructing the optical parameters of the radiative transfer equation (RTE) from boundary measurements in the diffusion limit. In the diffusive regime (the Knudsen number $\mathsf{Kn} \ll 1$), the forward problem for the stationary RTE is well approximated by an elliptic equation. However, the connection between the inverse problem for the RTE and the inverse problem for the elliptic equation has not been fully developed. This problem is particularly interesting because the former one is mildly ill-posed, with a Lipschitz type stability estimate, while the latter is well known to be severely ill-posed with a logarithmic type stability estimate. In this paper, we derive stability estimates for the inverse problem for RTE and examine its dependence on Kn. We show that the stability is Lipschitz in all regimes, but the coefficient deteriorates as $e^{\frac{1}{\mathsf{Kn}}}$, making the inverse problem of RTE severely ill-posed when Kn is small. In this way we connect the two inverse problems. Numerical results agree with the analysis of worsening stability as the Knudsen number gets smaller.

Key words. radiative transfer equation, Knudsen number, stability estimate

AMS subject classifications. 35R30, 65M32

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1. Introduction.

Optical tomography (OT) is a technique that uses low-energy visible or near-infrared light in the wavelength region (650nm $\sim$ 900nm) to illuminate highly scattering media [3]. In OT, based on measurements of scattered and transmitted light intensities on the surface of the medium, a reconstruction of the spatial distribution of the optical properties, for instance, absorption coefficient, σ_a, and scattering coefficient, σ_s, inside the medium is attempted. OT has potential applications to a variety of science and engineering fields, including oceanography, atmospheric science, astronomy, and neutron physics [33]. More recently, OT has found an application to medical imaging, and this application has received considerable attraction. In particular, visible or near-infrared light is sent into tissues, and then one can distinguish between healthy and unhealthy tissues from the reconstructed optical parameters.

However, the problem has not been fully understood mathematically. In fact, there are a variety of forward models for describing photon propagation. The two widely applied models are the radiative transfer equation (RTE, also known as the linear Boltzmann equation) and the diffusion equation (DE). What is intriguing here is that these two models are good approximations to each other in the diffusion regime

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where the Knudsen number (Kn) in the RTE is small, while the corresponding inverse problems are proved to be mildly ill-posed and severely ill-posed, respectively. The aim of this article is to study the connection between these two models in the inverse problem setting. More precisely we study the stability estimate of the parameter for the RTE, and make its dependence on Kn explicit. We will show that despite the fact that the stability is Hölder-like, its coefficient blows up in an exponential fashion for small Kn. The derived estimate provides evidence that reveals the severe ill-posedness could occur when the RTE is in the diffusion regime.

We now give a brief review of both models.

1.1. RTE and its inverse problem. The widely applied equation in optical imaging is the RTE:

$$\begin{cases} v\cdot\nabla_x f(x,v)+(\sigma_a(x)+\sigma_s(x))f(x,v)-\sigma_s(x)\int_V p(v',v)f(x,v')dv'=0,\\ f|_{\Gamma_-}=\text{given data}, \end{cases} \tag{1.1}$$

which models photon transport in tissues at the position $x\in\Omega$ in the direction $v\in V$ [3, 4, 26]. In (1.1), $f(x,v)$ is defined on the phase space, and it represents the density of particles at position x with velocity v in an open set V. Moreover, σ_a and σ_s are two optical parameters representing the absorption coefficient and the scattering coefficient, respectively. In particular, these optical parameters model how likely a photon particle is absorbed or scattered by the media. The scattering phase function is

$$k(x,v,v')=\sigma_s(x)p(v,v')$$

that defines the probability that during a scattering event, a photon from direction v' is scattered in the direction v at the point x. We also define the total absorption coefficient by $\sigma=\sigma_a+\sigma_s$. It measures how likely a photon with velocity v disappears (from either being absorbed or scattered).

We consider the physical domain Ω, which is a subset of $\mathbb{R}^n$, $n=2,3$. Suppose that Ω is a bounded open convex set with C^1 boundary $\partial\Omega$. Let V be the velocity domain which is an open set in $\mathbb{R}^n$. We assume that there are constants M_1 and M_2 such that

$$0<M_1<|v|<M_2 \tag{1.2}$$

for all $v\in V$. In addition, Γ_+ and Γ_- are used to denote the coordinates on the physical boundary associated with the outgoing and incoming velocities, respectively,

$$\Gamma_\pm=\{(x,v)\in\partial\Omega\times V:\pm\, n_x\cdot v>0\},$$

where n_x is the unit outer normal to $\partial\Omega$ at the point $x\in\partial\Omega$. The boundary condition, therefore, is placed on Γ_-.

The inverse problem for the transport equation amounts to reconstructing the unknown optical parameters $\sigma_a+\sigma_s$ and k from the boundary measurement. Specifically, the boundary data we utilize is the *albedo* operator:

$$\mathcal{A}: f|_{\Gamma_-}\to f|_{\Gamma_+},$$

which maps from the incoming photon density $f|_{\Gamma_-}$ to the outgoing photon density $f|_{\Gamma_+}$. Thus, the inverse transport problem is to reconstruct $\sigma_a+\sigma_s$ and k from the entire albedo map.

The theoretical approach on the reconstruction of the optical parameters (σ, k) is based on the singular decomposition of the Schwartz kernel $\alpha = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3$ of the albedo operators $\mathcal{A}$, where $\mathcal{A}_j$, $j = 1, 2, 3$, are described in section 2.2. For detailed discussion, we refer the reader to [14, 16, 35]. In the remainder of this subsection, we briefly discuss how this approach enables the reconstruction of the coefficients σ and k through the study of the kernel.

We address the properties of the kernel $\mathcal{A}_j$ for both the time-independent problem and the time-dependent problem. For the time-independent problem in dimension $n \geq 3$, one can observe that $\mathcal{A}_1$ and $\mathcal{A}_2$ are delta functions, while $\mathcal{A}_3$ is a locally L^1 function. Thus, $\mathcal{A}_3$ can be distinguished from $\mathcal{A}_1 + \mathcal{A}_2$. In addition, $\mathcal{A}_1$ can be separated from $\mathcal{A}_1 + \mathcal{A}_2$ since $\mathcal{A}_1$ and $\mathcal{A}_2$ have different degrees of singularities. From the information of $\mathcal{A}_1$, the parameter σ can be reconstructed. Moreover, one can further recover k from $\mathcal{A}_2$. As for the time-dependent problem, a similar procedure can be used to recover both σ and k for any dimension $n \geq 2$ without additional assumption on k. However, in the two-dimensional case for the time-independent problem, since $\mathcal{A}_2$ is a locally L^1 function as well, it cannot be distinguished from $\mathcal{A}_3$. Therefore, the same approach does not work for recovering k. However, $\mathcal{A}_1$ and $\mathcal{A}_2$ are still distinguishable; as a result, one can still recover σ for the stationary case in two dimensions.

In the absence of the time variable, it is the inverse stationary transport problem which contains less data compared to the time-dependent transport problem. This inverse stationary transport problem is overdetermined in dimension $n \geq 3$ since the kernel of $\mathcal{A}$ depends on higher dimensions than those on which the parameters σ and k depend. The unique result for both optical parameters σ and k was studied in [15, 16]. The associated stability estimate was derived in [5, 6, 7, 38] in dimension $n \geq 3$. However, in $n = 2$, the problem is only formally determined for the reconstruction of k, but it is still overdetermined for the reconstruction of σ. It was investigated in [36] that when a smallness assumption on k is imposed, both coefficients σ and k can be uniquely determined. Finally, we remark that the unique determination of σ and k for the time-dependent transport equation was studied in [13, 14] for any dimension $n \geq 2$ without assuming any smallness assumption on k.

1.2. Diffusion equation and its inverse problem. Depending on the relation between the scattering coefficient σ_s and the absorption coefficient σ_a, the RTE can sometimes be approximated by the diffusion approximation. More specifically, in scatter dominated materials the diffusion approximation remains valid, but in materials where σ_a dominates or is comparable to σ_s, the diffusion approximation is not a suitable model. The first case is seen in breast tissue where σ_s is much larger than σ_a at appropriate wavelengths (650nm $\sim$ 900nm) [18]. More work on breast image studies can be found in [17, 19]. However, when the absorption coefficient of the medium is similar to the scattering coefficient, the diffusion approximation might not be a good approximation to describe the photo migration in biological tissues. For example, in the blood vessels or organs with a high blood perfusion, such as in the liver, the approximation does not hold at any wavelength. We refer the interested reader to [20], for instance.

Assume that the scattering effect dominates; then the diffusion equation is modeled by

$$-C\nabla_x \cdot (\sigma_s^{-1} \nabla_x \Phi(x)) + \sigma_a \Phi(x) = 0 \tag{1.3}$$

with a constant C. Note that the diffusive medium takes the reciprocal of σ_s, and the photon intensity is defined by

$$\Phi(x) = \int_V f(x,v)\mathrm{d}v\,.$$

We refer to [4] for a detailed discussion on the transport equation in the diffusive regime, and only cite the results here.

Now we consider the strong scattering case. We define the total absorption coefficient and the scattering coefficient in the diffusion limit:

$$\sigma_{\mathsf{Kn}}(x) = \mathsf{Kn}\sigma_a + \mathsf{Kn}^{-1}\sigma_s \quad \text{for } 0 < \mathsf{Kn} \ll 1 \tag{1.4}$$

and

$$k = \mathsf{Kn}^{-1}\sigma_s(x).$$

In order to make our approach and idea clear, we set the phase function $p(v',v)$ to be 1. For a more general p which is not singular, since the phase function does not affect the estimate of the total absorption coefficient, similar stability estimates as in Theorem 1.2 are expected to be valid as well. Hereafter, we replace σ by σ_{Kn} and k by $\mathsf{Kn}^{-1}\sigma_s$ in (1.1), and we have the RTE in the diffusion regime:

$$\begin{cases} v\cdot\nabla_x f(x,v) + \sigma_{\mathsf{Kn}}(x) f(x,v) - \mathsf{Kn}^{-1}\sigma_s(x)\int_V f(x,v')dv' = 0 & \text{in } \Omega\times V, \\ f|_{\Gamma_-} = f_-. \end{cases} \tag{1.5}$$

The following result can be proved from [10, 11, 28, 29, 39].

THEOREM 1.1. *Suppose that f solves (1.5). As $\mathsf{Kn}\to 0$, $f(x,v)$ converges to $\rho(x)$, where $\rho(x)$ solves the diffusion equation:*

$$C\nabla_x\cdot(\sigma_s^{-1}\nabla_x\rho) - \sigma_a\rho = 0\,. \tag{1.6}$$

Here C is a constant depending on the dimension of the problem. The boundary condition is determined by

$$\rho|_{\partial\Omega} = \xi_f \tag{1.7}$$

with $\xi_f(x_0) = f^l_{z\to\infty}$, where f^l solves the boundary layer equation:

$$v\partial_z f^l = \sigma_s\left(\int f^l(z,v')\mathrm{d}v' - f^l\right), \quad z\in[0,\infty) \quad \textit{with} \quad f^l|_{z=0} = f_-(x_0,v).$$

Moreover, one has

1. $\int (v\cdot n_{x_0}) f(x_0,v)\mathrm{d}v = \mathsf{Kn}\sigma_s^{-1}\partial_{n_{x_0}}\rho(x_0) + \mathcal{O}(\mathsf{Kn}^2)$, *and*
2. *if $f_-(x,v) = f_-(x)$ is independent of v for all $x\in\partial\Omega$, then $\rho|_{\partial\Omega} = \xi_f = f_-$.*

Remark 1.1. We define the averaged albedo operator for the problem (1.5) by

$$\mathcal{A}_{ave}[f_-] = \mathsf{Kn}^{-1}\int (v\cdot n_x) f(x,v)\mathrm{d}v\,,$$

and define the Dirichlet-to-Neumann (DtN) map for the diffusion equation (1.6) by

$$\Lambda[\rho|_{\partial\Omega}] = \partial_{n_x}\rho(x)\,.$$

Assume that the incoming boundary condition f_- is homogeneous in v. According to Theorem 1.1, $f_-(x_0) = \rho(x_0)$ on $\partial\Omega$, and the averaged albedo operator converges to the DtN map, namely,

$$\mathcal{A}_{ave}[f_-] - \Lambda[\rho|_{\partial\Omega}] = \mathsf{Kn}^{-1}\int (v\cdot n_x) f(x,v)\mathrm{d}v - \partial_{n_{x_0}}\rho(x_0) = \mathcal{O}(\mathsf{Kn})\,. \tag{1.8}$$

It is well known that the inverse problem for the elliptic equation (1.6), that is, using the DtN map to recover the coefficients in (1.6), is severely ill-posed. In particular, it has a logarithmic type of stability estimate. This kind of estimate was first derived by Alessandrini in [1] and was shown to be optimal in [32]. For reviews of the stability issue and Calderón's problem, we refer the reader to [2, 37]. In contrast to the inverse problem for the elliptic equation, the inverse problem for the RTE (1.5) has a Hölder type stability; see [5, 6, 7, 38].

1.3. Main result. In this article, we are interested in bridging the stability estimates for these two inverse problems. We are motivated by the study of increasing stability behavior for several elliptic inverse problems when the frequency gets higher. It is known that the logarithmic stability makes reconstruction algorithms challenging since a small error in the data could be magnified exponentially in the numerical reconstruction. The research on increasing stability therefore arises from the desire to design a more reliable reconstruction algorithm. Its central idea is to obtain stability estimates that contain two parts: one is Hölder, the other is logarithmic, and their associated coefficients explicitly depend on certain parameters (frequency) in the forward model. With the frequency chosen in a suitable range, one part of the estimate dominates the other, which leads to increasing stability or stability deterioration. This type of problem has been addressed in [21, 23, 24, 25, 27, 30, 34] in different contexts. We also refer the interested reader to the book [22] for more detailed discussion on the study of increasing stability in different problems. For the reconstruction of optical parameters in RTE, particularly in [8, 9] the authors studied the stability of the inversion with respect to the modulation frequency in a time-harmonic setting for RTE, and found that the increasing of the frequency brings more details in the recovery. Without the time dependence, the stability deterioration of the linearized problem was investigated in [12]. Unknown to us during our study for the current work, parallel research was conducted in [40], where the authors showed that the instability is also of exponential type.

In this paper, we investigate the stability estimate for the RTE based on the analysis of the albedo operator and, moreover, we trace its explicit dependence on Kn. Assume that the media $(\sigma_{\mathsf{Kn}}, \mathsf{Kn}^{-1}\sigma_s)$ are admissible (made clear in section 2.2) and satisfy

$$\|\tau\sigma_{\mathsf{Kn}}\|_{L^\infty} < \infty, \quad \left\|\tau\int_V \mathsf{Kn}^{-1}\sigma_s(x)dv\right\|_{L^\infty} < \infty, \quad \text{and} \quad \sigma_{\mathsf{Kn}} \geq \int_V \mathsf{Kn}^{-1}\sigma_s(x)dv \tag{1.9}$$

for almost everywhere (a.e.) $x \in \Omega$ which ensures the boundary value problem (1.5) being well-posed. We define the space

$$\mathcal{P} = \{u \in H^{n/2+r'}(\Omega):\ u \geq 0,\ \mathrm{supp}(u) \subset \Omega,\ \|u\|_{H^{n/2+r'}(\mathbb{R}^n)} \leq M_3\} \tag{1.10}$$

for $n = 2,\ 3$ and for some $r' > 0$. Here $\mathrm{supp}(u)$ denotes the compact support of a function u.

Our main result is stated as follows, and its proof will be given in section 3.

THEOREM 1.2 (stability estimate with explicit Kn dependence). *Let Ω be a bounded open convex set in $\mathbb{R}^n$, $n = 2, 3$ with C^1 boundary $\partial\Omega$, and let $0 < \mathsf{Kn} < 1$. Suppose the assumption (1.9) holds and the functions σ_a, σ_s, $\tilde\sigma_a$, and $\tilde\sigma_s$ are in $\mathcal{P}$. We denote $\mathcal{A}$ and $\tilde{\mathcal{A}}$ as albedo operators associated with media pairs $(\sigma_{\mathsf{Kn}}, \mathsf{Kn}^{-1}\sigma_s)$ and $(\tilde\sigma_{\mathsf{Kn}}, \mathsf{Kn}^{-1}\tilde\sigma_s)$, respectively. Then for some $\theta \in (0,1)$, there exists a constant C,* *independent of* Kn*, such that the estimate*

$$\|\sigma_{\mathsf{Kn}} - \tilde{\sigma}_{\mathsf{Kn}}\|_{L^\infty} \leq C\mathsf{Kn}^{-1+\theta} e^{C\theta\mathsf{Kn}^{-1}} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\theta$$

holds, where $\|\cdot\|_*$ *is the operator norm from* $L^1(\Gamma_-, d\xi)$ *to* $L^1(\Gamma_+, d\xi)$*.*

Moreover, if $\mathsf{Kn} < |\log(\|\mathcal{A} - \tilde{\mathcal{A}}\|_*)|^{-\alpha}$ *for some* $\alpha > 0$*, then*

$$\|\sigma_s - \tilde{\sigma}_s\|_{L^\infty} \leq C\mathsf{Kn}|\log(\|\mathcal{A} - \tilde{\mathcal{A}}\|_*)|^{-\alpha} + C\mathsf{Kn}^\theta e^{C\theta\mathsf{Kn}^{-1}} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\theta \tag{1.11}$$

and

$$\|\sigma_a - \tilde{\sigma}_a\|_{L^\infty} \leq C\mathsf{Kn}^{-3}|\log(\|\mathcal{A} - \tilde{\mathcal{A}}\|_*)|^{-\alpha} + C\mathsf{Kn}^{-2+\theta} e^{C\theta\mathsf{Kn}^{-1}} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\theta. \tag{1.12}$$

This theorem shows that the exponential component occurs in the stability estimates (1.11) and (1.12) within a certain range of Kn. This provides certain evidence which shows that the Hölder stability for the RTE can be connected to the logarithmic stability for the Caldéron problem. For more detailed discussion, see remarks at the end of section 3.

We comment that in [40], the authors investigated the "instability" (instead of "stability") of the problem and its dependence on Kn. In some sense, their work gives a $\geq$ sign in (1.11) and (1.12) above. Their work, together with ours, forms a complete picture.

This paper is organized as follows. In section 2, we discuss preliminaries and state several known results about the albedo operator decomposition. Section 3 is devoted to the study of the stability estimate whose coefficient explicitly depends on the Knudsen number. Numerical examples are provided in section 4 that confirm both the Hölder stability and the logarithmic ill-posedness for small Kn, and therefore the numerical experiments are in agreement with the statements in Theorem 1.2.

2. Preliminaries.

In this section, we recall several function spaces and introduce notation, as well as some known results. They are relevant in our setup and in the reconstruction of the optical parameters discussed in section 3.

2.1. Function spaces.

We define the Sobolev spaces $H^s(\mathbb{R}^n)$ in the whole space by

$$H^s(\mathbb{R}^n) = \left\{u \in \mathcal{S}' : \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^n)}\right\},$$

where $\langle D\rangle^s u := \mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}u)$. Here $\mathcal{F}u$ denotes the Fourier transform of u and $\mathcal{S}'$ is the dual of the Schwartz space $\mathcal{S}$. In addition, for an open set U in $\mathbb{R}^n$, we define the class of functions in $H^s(\mathbb{R}^n)$ which is restricted in U by

$$H^s(U) = \{u|_U : u \in H^s(\mathbb{R}^n)\}.$$

Moreover, we define the measure on the incoming and outgoing coordinates $\Gamma_\pm$ by

$$d\xi(x,v) = |n_x \cdot v| d\mu(x) dv \tag{2.1}$$

with the measure $d\mu(x)$ defined on the boundary $\partial\Omega$. We also denote $L^1(\Gamma_\pm, d\xi)$ to be the space consisting of functions u satisfying

$$\int_{\Gamma_\pm} |u(x,v)| d\xi(x,v) < \infty.$$

2.2. Kernel of the albedo operator. We consider the boundary value problem with Dirichlet boundary condition for the stationary transport equation:

$$\begin{cases} v\cdot\nabla_x f(x,v)+\sigma(x,v)f(x,v)-\int_V k(x,v',v)f(x,v')dv'=0 & \text{in } \Omega\times V,\\ f|_{\Gamma_-}=f_-. \end{cases}$$

The pair (σ,k) is called *admissible* if

$$0\le\sigma\in L^\infty(\Omega\times V) \tag{2.2}$$

and

$$0\le k(x,v',\cdot)\in L^1(V) \tag{2.3}$$

for a.e. $(x,v')\in\Omega\times V$. Moreover, we define the scattering cross sections by $\int_V k(x, v',v)dv$, which is in $L^\infty(\Omega\times V)$. The collected data is defined by the albedo operator

$$\mathcal{A}: f|_{\Gamma_-}\to f|_{\Gamma_+},$$

which maps the incoming Dirichlet type boundary condition into the outgoing one. In particular, $\mathcal{A}$ is a bounded operator from $L^1(\Gamma_-,d\xi)$ to $L^1(\Gamma_+,d\xi)$, as shown in [16].

In the diffusion regime, we consider optical parameters $(\sigma_{\mathsf{Kn}},\mathsf{Kn}^{-1}\sigma_s)$, instead of (σ,k) as in section 1.3. Assume that $(\sigma_{\mathsf{Kn}},\mathsf{Kn}^{-1}\sigma_s)$ is also admissible. From [13, 14, 16], it was shown that the albedo operator $\mathcal{A}$ is bounded from $L^1(\Gamma_-,d\xi)$ to $L^1(\Gamma_+,d\xi)$ equipped with the kernel $\alpha(x,v,x',v')=(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3)(x,v,x',v')$, where

$$\mathcal{A}_1(x,v,x',v')=e^{-\int_0^{\tau_-(x,v)}\sigma_{\mathsf{Kn}}(x-tv)dt}\delta_{x-\tau_-(x,v)v}(x')\delta(v-v'), \tag{2.4}$$

$$\begin{aligned}\mathcal{A}_2(x,v,x',v')=-\int_0^{\tau_-(x,v)} & e^{-\int_0^\eta\sigma_{\mathsf{Kn}}(x-tv)dt-\int_0^{\tau_-(x-\eta v,v')}\sigma_{\mathsf{Kn}}(x-\eta v-tv')dt}\\ & k(x-\eta v,v',v)\delta_{x-\eta v-\tau_-(x-\eta v,v')v'}(x')d\eta,\end{aligned} \tag{2.5}$$

and

$$|n_{x'}\cdot v'|^{-1}\mathcal{A}_3(x,v,x',v')\in L^\infty(\Gamma_-;L^1(\Gamma_+,d\xi)). \tag{2.6}$$

Here $d\xi(x,v)$ is defined in (2.1). In addition, $\delta(x)$ is the delta function on $\mathbb{R}^n$ and $\delta_y(x)$ is the delta function on $\partial\Omega$ defined by

$$(\delta_y,h)=h(y)$$

for any $h\in C_c^\infty(\mathbb{R}^n)$. The travel time is denoted by

$$\tau_\pm(x,v)=\min\{t\ge0:(x\pm tv,v)\in\Gamma_\pm\}.$$

Notice that the kernel $\mathcal{A}_1$ is a singular distribution supported on the surface $x'=x-\tau_-(x,v)v$ and $v=v'$. One can apply the different degrees of singularities of $\mathcal{A}_j$, $j=1,2,3$, to distinguish $\mathcal{A}_1$ from the whole kernel α. Thus, the information of σ_{Kn} can be extracted from $\mathcal{A}_1$. More precisely, the X-ray transform (defined in (3.10)) of σ_{Kn} will be first recovered from $\mathcal{A}_1$. Finally, based on this, we could derive the stability estimate for σ_{Kn} with coefficients that depend explicitly on Kn; see section 3.

We recall Lemmas 2.1 and 2.2 whose proofs can be found in [16] and [38], respectively.

LEMMA 2.1. *Let $f \in L^1(\Omega \times V)$. Then*

$$\int_{\Omega\times V} f(x,v)dxdv = \int_{\Gamma_\mp} \int_0^{\tau_\pm(x',v)} f(x' \pm tv, v)dtd\xi(x',v)\,.$$

LEMMA 2.2. *Let $f \in L^1(\Gamma_-, d\xi)$. Then*

$$\int_{\Gamma_+} f(x - \tau_-(x,v)v, v)d\xi(x,v) = \int_{\Gamma_-} f(x',v)d\xi(x',v)\,.$$

These identities will play a crucial role in the derivation of the stability estimate for the optical parameters. In particular, Lemma 2.2 implies that integrals on the space Γ_+ and the space Γ_- are the same under a suitable change of variables. It will be applied in Lemma 3.1 in order to transform the integral over the outgoing space Γ_+ into the integral over the incoming space Γ_- such that the X-ray transform of σ_{Kn} can be recovered from the leading kernel $\mathcal{A}_1$. As for Lemma 2.1, it gives a way to compute the integral over $\Omega \times V$ by using the line and surface integrals, and vice versa.

3. Analysis of the Knudsen number. In this section, we study the stability estimate of the absorption and of the scattering coefficient and, in particular, we keep track of their dependence on the Kn. We start by analyzing the total absorption coefficient $\sigma_{\mathsf{Kn}} = \mathsf{Kn}\sigma_a + \mathsf{Kn}^{-1}\sigma_s$ defined in (1.4). The analytic technique is based on [15, 16, 38] with suitable adjustments to our setting.

3.1. Stability estimate of the total absorption coefficient $\boldsymbol{\sigma_{\mathsf{Kn}}}$. Assuming that the function $\psi \in C_0^\infty(\mathbb{R}^n)$ satisfies $0 \le \psi \le 1$, then $\psi(0) = 1$ and $\int \psi dx = 1$. Let $(x_0', v_0') \in \Gamma_-$ and $\varepsilon > 0$. We denote the functions

$$\psi^\varepsilon_{v_0'}(v) = \varepsilon^{-n}\psi\left(\frac{v - v_0'}{\varepsilon}\right).$$

We also choose functions $\phi^\varepsilon_{x_0'}$ in spatial dimensions such that $0 \le \phi^\varepsilon_{x_0'}(x) \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp} \phi^\varepsilon_{x_0'}(x) \in B^\varepsilon(x_0') \cap \partial\Omega$. Moreover, they satisfy $\int_{\partial\Omega} \phi^\varepsilon_{x_0'}(x)d\mu(x) = 1$ and

$$\lim_{\varepsilon\to 0} \int_{\partial\Omega} f(x)\phi^\varepsilon_{x_0'}(x)d\mu(x) = f(x_0')$$

for any function f in $C^0(\partial\Omega)$. For any point (x_0', v_0') in Γ_-, if $\varepsilon > 0$ is sufficiently small, then one has

$$\operatorname{supp} \phi^\varepsilon_{x_0'}(x') \times \operatorname{supp} \psi^\varepsilon_{v_0'}(v') \subset \Gamma_-.$$

Next, we choose the smooth cut off function $f^\varepsilon_{x_0',v_0'}$ on $\partial\Omega$ and on the velocity space by

$$f^\varepsilon_{x_0',v_0'}(x',v') = |n_{x'} \cdot v'|^{-1}\phi^\varepsilon_{x_0'}(x')\psi^\varepsilon_{v_0'}(v'). \tag{3.1}$$

From a direct computation, one has $f^\varepsilon_{x_0',v_0'} \in L^1(\Gamma_-, d\xi)$. In particular, $f^\varepsilon_{x_0',v_0'}$ satisfies

$$\|f^\varepsilon_{x_0',v_0'}\|_{L^1(\Gamma_-,d\xi)} = \int_{\Gamma_-} |f^\varepsilon_{x_0',v_0'}(x',v')||n_{x'} \cdot v'|d\mu(x')dv' = 1. \tag{3.2}$$

Before studying the kernel, for any fixed point $(x_0', v_0') \in \Gamma_-$, we define another cut off function on Γ_+ by

$$\tilde{\chi}^\varepsilon(x,v) := \chi^\varepsilon(x - \tau_-(x,v)v, v) = \chi^{1,\varepsilon}_{x_0'}(x - \tau_-(x,v)v)\chi^{2,\varepsilon}_{v_0'}(v)$$

for any (x,v) in Γ_+, where $\chi^{1,\varepsilon}$ and $\chi^{2,\varepsilon}$ satisfy

$$\begin{cases} \chi^{1,\varepsilon}(x) = 1 & \text{in } B^\varepsilon(x_0') \cap \partial\Omega, \\ \chi^{1,\varepsilon}(x) = 0 & \text{in } \mathbb{R}^n \setminus (B^\varepsilon(x_0') \cap \partial\Omega) \end{cases}$$

and

$$\begin{cases} \chi^{2,\varepsilon}(v) = 1 & \text{in } \operatorname{supp} \psi^\varepsilon_{v_0'}(v), \\ \chi^{2,\varepsilon}(v) = 0 & \text{in } V \setminus \operatorname{supp} \psi^\varepsilon_{v_0'}(v). \end{cases}$$

The main goal of this section is to extract the information of $\sigma_{\mathsf{Kn}} - \tilde{\sigma}_{\mathsf{Kn}}$ from the measurements $\mathcal{A} - \tilde{\mathcal{A}}$. Let α and $\tilde{\alpha}$ be the distribution kernel for $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively. We apply the cut off function on the albedo operator and then estimate the function

$$\begin{aligned} &\tilde{\chi}^\varepsilon(\mathcal{A} - \tilde{\mathcal{A}}) f^\varepsilon_{x_0',v_0'}(x,v) \\ &= \tilde{\chi}^\varepsilon(x,v) \int_{\Gamma_-} (\alpha(x,v,x',v') - \tilde{\alpha}(x,v,x',v')) f^\varepsilon_{x_0',v_0'}(x',v') d\mu(x') dv' \end{aligned}$$

for any point $(x,v) \in \Gamma_+$. More specifically, we will estimate each term in the right-hand side of (3.3) which corresponds to $\mathcal{A}_j$, $j = 1,2,3$, respectively,

$$\begin{aligned} &\|\tilde{\chi}^\varepsilon(\mathcal{A} - \tilde{\mathcal{A}}) f^\varepsilon_{x_0',v_0'}\|_{L^1(\Gamma_+, d\xi)} \\ &= \left\| \sum_{j=1}^{3} \tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_j - \tilde{\mathcal{A}}_j) f^\varepsilon_{x_0',v_0'}(x',v') d\mu(x') dv' \right\|_{L^1(\Gamma_+, d\xi)}. \end{aligned} \tag{3.3}$$

We start by considering the following estimate.

Lemma 3.1. *For $\varepsilon > 0$, let $f^\varepsilon_{x_0',v_0'}$ be the function defined in* (3.1). *Then the following identity holds:*

$$\begin{aligned} &\lim_{\varepsilon \to 0} \left\| \tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_1 - \tilde{\mathcal{A}}_1) f^\varepsilon_{x_0',v_0'}(x',v') d\mu(x') dv' \right\|_{L^1(\Gamma_+, d\xi)} \\ &= \left| e^{-\int_{\mathbb{R}} \sigma_{\mathsf{Kn}}(x_0' + t v_0') dt} - e^{-\int_{\mathbb{R}} \tilde{\sigma}_{\mathsf{Kn}}(x_0' + t v_0') dt} \right|. \end{aligned}$$

Proof. From the definition of the kernels A_1 and $\tilde{A}_1$, one has

$$\begin{aligned} &\int_{\Gamma_-} (\mathcal{A}_1 - \tilde{\mathcal{A}}_1)(x,v,x',v') f^\varepsilon_{x_0',v_0'}(x',v') d\mu(x') dv' \\ &= \left(e^{-\int_0^{\tau_-(x,v)} \sigma_{\mathsf{Kn}}(x - tv) dt} - e^{-\int_0^{\tau_-(x,v)} \tilde{\sigma}_{\mathsf{Kn}}(x - tv) dt} \right) f^\varepsilon_{x_0',v_0'}(x - \tau_-(x,v)v, v). \end{aligned}$$

Since σ_{Kn} and $\tilde{\sigma}_{\mathsf{Kn}}$ are supported in Ω, the integration range can be extended to the whole space $\mathbb{R}$. Thus, we obtain

$$\begin{aligned}
&\left\|\tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_1 - \tilde{\mathcal{A}}_1) f^\varepsilon_{x_0',v_0'}(x',v')d\mu(x')dv'\right\|_{L^1(\Gamma_+,d\xi)} \\
&= \int_{\Gamma_+} \tilde{\chi}^\varepsilon(x,v) \left| e^{-\int_{\mathbb{R}} \sigma_{\mathsf{Kn}}(x-\tau_-(x,v)v+tv)dt} - e^{-\int_{\mathbb{R}} \tilde{\sigma}_{\mathsf{Kn}}(x-\tau_-(x,v)v+tv)dt} \right| \\
&\qquad f^\varepsilon_{x_0',v_0'}(x-\tau_-(x,v)v,v)d\xi(x,v) \\
&= \int_{\Gamma_-} \chi^\varepsilon(y,v) \left| e^{-\int_{\mathbb{R}} \sigma_{\mathsf{Kn}}(y+tv)dt} - e^{-\int_{\mathbb{R}} \tilde{\sigma}_{\mathsf{Kn}}(y+tv)dt} \right| f^\varepsilon_{x_0',v_0'}(y,v)d\xi(y,v). \qquad (3.4)
\end{aligned}$$

Here the last identity holds by applying Lemma 2.2 and the fact that $f^\varepsilon_{x_0',v_0'} \in L^1(\Gamma_-,d\xi)$. Moreover, from the definition of χ^ε, it is clear that χ^ε is compactly supported and $\chi^\varepsilon = 1$ in $(B^\varepsilon(x_0') \cap \partial\Omega) \times \operatorname{supp} \psi^\varepsilon_{v_0'}$. We apply the properties of the function $f^\varepsilon_{x_0',v_0'}$ with (3.2) and then, by taking the limit $\varepsilon \to 0$ on the identity (3.4), we conclude that

$$\begin{aligned}
&\left\|\tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_1 - \tilde{\mathcal{A}}_1) f^\varepsilon_{x_0',v_0'}(x',v')d\mu(x')dv'\right\|_{L^1(\Gamma_+,d\xi)} \\
&= \int_{\Gamma_-} \left| e^{-\int_{\mathbb{R}} \sigma_{\mathsf{Kn}}(y+tv)dt} - e^{-\int_{\mathbb{R}} \tilde{\sigma}_{\mathsf{Kn}}(y+tv)dt} \right| f^\varepsilon_{x_0',v_0'}(y,v)d\xi(y,v) \\
&\to \left| e^{-\int_{\mathbb{R}} \sigma_{\mathsf{Kn}}(x_0'+tv_0')dt} - e^{-\int_{\mathbb{R}} \tilde{\sigma}_{\mathsf{Kn}}(x_0'+tv_0')dt} \right|.
\end{aligned}$$

This finishes the proof. ∎

For the remaining two terms in (3.3), we have the following identities.

LEMMA 3.2. *For $\varepsilon > 0$, $f^\varepsilon_{x_0',v_0'}$ is defined in* (3.1). *Then*

$$\lim_{\varepsilon\to 0} \left\|\tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_j - \tilde{\mathcal{A}}_j) f^\varepsilon_{x_0',v_0'}(x',v')d\mu(x')dv'\right\|_{L^1(\Gamma_+,d\xi)} = 0, \quad j = 2,3.$$

Proof. We first study the case $j = 2$. From the definition of the kernel $\mathcal{A}_2$ in (2.5), the delta function $\delta_{x-\eta v-\tau_-(x-\eta v,v')v'}(x')$ acts on the function $f^\varepsilon_{x_0',v_0'}(x',v')$. This takes the value $f^\varepsilon_{x_0',v_0'}(x-\tau_-(x-\eta v,v')v'-\eta v,v')$. Therefore, one obtains that

$$\begin{aligned}
&\left\|\tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_2 - \tilde{\mathcal{A}}_2) f^\varepsilon_{x_0',v_0'}(x',v')d\mu(x')dv'\right\|_{L^1(\Gamma_+,d\xi)} \\
&= \int_{\Gamma_+} \tilde{\chi}^\varepsilon(x,v) \bigg| \int_V \int_0^{\tau_-(x,v)} (\Gamma - \tilde{\Gamma})(x,v,x',v') \\
&\qquad f^\varepsilon_{x_0',v_0'}(x-\tau_-(x-\eta v,v')v'-\eta v,v')d\eta dv' \bigg| d\xi(x,v),
\end{aligned}$$

where we denote

$$\Gamma(x,v,x',v') = \mathsf{Kn}^{-1} e^{-\int_0^\eta \sigma_{\mathsf{Kn}}(x-tv)dt - \int_0^{\tau_-(x-\eta v,v')} \sigma_{\mathsf{Kn}}(x-\eta v-tv')dt} \sigma_s(x-\eta v)$$

and

$$\tilde{\Gamma}(x,v,x',v') = \mathsf{Kn}^{-1} e^{-\int_0^\eta \tilde{\sigma}_{\mathsf{Kn}}(x-tv)dt - \int_0^{\tau_-(x-\eta v,v')} \tilde{\sigma}_{\mathsf{Kn}}(x-\eta v - tv')dt} \tilde{\sigma}_s(x-\eta v).$$

Note that since σ_s and $\tilde{\sigma}_s$ are nonnegative, one gets

$$|\Gamma| \leq \mathsf{Kn}^{-1}\sigma_s(x-\eta v), \quad |\tilde{\Gamma}| \leq \mathsf{Kn}^{-1}\tilde{\sigma}_s(x-\eta v).$$

We then interchange the integration order by using Fubini's theorem and Lemma 2.1; therefore, we have

$$\begin{aligned}
&\left\| \tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_2 - \tilde{\mathcal{A}}_2) f^\varepsilon_{x_0',v_0'}(x',v') d\mu(x') dv' \right\|_{L^1(\Gamma_+,d\xi)} \\
&\leq \int_{\Gamma_+} \int_V \int_0^{\tau_-(x,v)} \chi^{2,\varepsilon}(v) \mathsf{Kn}^{-1} (\sigma_s + \tilde{\sigma}_s)(x-\eta v) \\
&\qquad f^\varepsilon_{x_0',v_0'}(x-\eta v - \tau_-(x-\eta v,v')v',v') d\eta dv' d\xi(x,v) \\
&= \int_V \int_{\Omega\times V} \chi^{2,\varepsilon}(v) \mathsf{Kn}^{-1}(\sigma_s+\tilde{\sigma}_s)(x) f^\varepsilon_{x_0',v_0'}(x-\tau_-(x,v')v',v') dx dv dv'.
\end{aligned} \tag{3.5}$$

Using Lemma 2.1 again and the bounded condition for v described in (1.2) leads to

$$\begin{aligned}
&\int_V \chi^{2,\varepsilon}(v) \Big(\int_{\Omega\times V} \mathsf{Kn}^{-1}(\sigma_s+\tilde{\sigma}_s)(x) f^\varepsilon_{x_0',v_0'}(x-\tau_-(x,v')v',v') dx dv' \Big) dv \\
&= \int_V \chi^{2,\varepsilon}(v) \Big(\int_{\Gamma_-} \int_0^{\tau_+(x',v')} \mathsf{Kn}^{-1}(\sigma_s+\tilde{\sigma}_s)(x'+tv') f^\varepsilon_{x_0',v_0'}(x',v') dt d\xi(x',v') \Big) dv \\
&\leq \frac{\mathrm{diam}(\Omega)}{M_1} \mathsf{Kn}^{-1} \|(\sigma_s+\tilde{\sigma}_s)\|_{L^\infty(\Omega)} \Big(\int_{\Gamma_-} f^\varepsilon_{x_0',v_0'}(x',v') d\xi(x',v') \Big) \Big(\int_{\mathrm{supp}\, \psi^\varepsilon_{v_0'}(v)} dv \Big).
\end{aligned} \tag{3.6}$$

In the last component of the above identity, the measure of $\mathrm{supp}\, \psi^\varepsilon_{v_0'}(v)$ goes to 0 when $\varepsilon \to 0$. It implies that the right-hand side of (3.6) converges to zero, and thus we obtain the conclusion of the lemma for the case $j = 2$.

Now we turn to the term with kernels $\mathcal{A}_3$ and $\tilde{\mathcal{A}}_3$. From (2.6), one has

$$|n_{x'} \cdot v'|^{-1} \mathcal{A}_3, \quad |n_{x'} \cdot v'|^{-1} \tilde{\mathcal{A}}_3 \in L^\infty(\Gamma_-; L^1(\Gamma_+, d\xi)).$$

Thus, the limit of

$$\begin{aligned}
&\left\| \tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_3 - \tilde{\mathcal{A}}_3) f^\varepsilon_{x_0',v_0'}(x',v') d\mu(x') dv' \right\|_{L^1(\Gamma_+,d\xi)} \\
&= \int_{\Gamma_+} \tilde{\chi}^\varepsilon(x,v) \left| \int_{\Gamma_-} (\mathcal{A}_3 - \tilde{\mathcal{A}}_3)(x,v,x',v') f^\varepsilon_{x_0',v_0'}(x',v') d\mu(x') dv' \right| d\xi(x,v) \\
&\leq \int_{\Gamma_+} \tilde{\chi}^\varepsilon(x,v) \int_{\Gamma_-} |n_{x'} \cdot v'|^{-1} |(\mathcal{A}_3 - \tilde{\mathcal{A}}_3)(x,v,x',v')| f^\varepsilon_{x_0',v_0'}(x',v') d\xi(x',v') d\xi(x,v) \\
&\leq \int_{\Gamma_+} \tilde{\chi}^\varepsilon(x,v) \sup_{(x',v')\in\Gamma_-} |n_{x'} \cdot v'|^{-1} |(\mathcal{A}_3 - \tilde{\mathcal{A}}_3)(x,v,x',v')| d\xi(x,v)
\end{aligned} \tag{3.7}$$

goes to 0 as $\varepsilon \to 0$ by applying the dominated convergence theorem and the fact that the measure of support of $\tilde{\chi}^\varepsilon$ converges to zero as $\varepsilon \to 0$. This completes the proof of this lemma. $\square$

From (3.3), we have the estimate for the term containing $\mathcal{A}_1 - \tilde{\mathcal{A}}_1$:

$$\begin{aligned}
&\left\| \tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_1 - \tilde{\mathcal{A}}_1) f^\varepsilon_{x_0', v_0'} d\mu(x') dv' \right\|_{L^1(\Gamma_+, d\xi)} \\
&\le \|\tilde{\chi}^\varepsilon (\mathcal{A} - \tilde{\mathcal{A}}) f^\varepsilon_{x_0', v_0'}\|_{L^1(\Gamma_+, d\xi)} + \sum_{j=2,3} \left\| \tilde{\chi}^\varepsilon \int_{\Gamma_-} (\mathcal{A}_j - \tilde{\mathcal{A}}_j) f^\varepsilon_{x_0', v_0'} d\mu(x') dv' \right\|_{L^1(\Gamma_+, d\xi)}.
\end{aligned}$$

When ε goes to 0, Lemmas 3.1 and 3.2 imply

$$\begin{aligned}
\left| e^{-\int_{\mathbb{R}} \sigma_{\mathsf{Kn}}(x_0' + t v_0') dt} - e^{-\int_{\mathbb{R}} \tilde{\sigma}_{\mathsf{Kn}}(x_0' + t v_0') dt} \right| &\le \lim_{\varepsilon \to 0} \|\tilde{\chi}^\varepsilon (\mathcal{A} - \tilde{\mathcal{A}}) f^\varepsilon_{x_0', v_0'}\|_{L^1(\Gamma_+, d\xi)} \\
&\le \|(\mathcal{A} - \tilde{\mathcal{A}}) f^\varepsilon_{x_0', v_0'}\|_{L^1(\Gamma_+, d\xi)} \\
&\le \|\mathcal{A} - \tilde{\mathcal{A}}\|_* \|f^\varepsilon_{x_0', v_0'}\|_{L^1(\Gamma_-, d\xi)} \\
&= \|\mathcal{A} - \tilde{\mathcal{A}}\|_*,
\end{aligned}$$

where we also use the fact that $\|f^\varepsilon_{x_0', v_0'}\|_{L^1(\Gamma_-, d\xi)} = 1$ in the last identity. Therefore, applying the mean value theorem on the left-hand side of the above inequalities, it follows that

$$\|\mathcal{A} - \tilde{\mathcal{A}}\|_* \ge e^{-\beta_{\mathsf{Kn}}} \left| \int_{\mathbb{R}} \sigma_{\mathsf{Kn}}(x_0' + t v_0') - \tilde{\sigma}_{\mathsf{Kn}}(x_0' + t v_0') dt \right|, \tag{3.8}$$

where one can deduce the constant bound β_{Kn} by using once again the boundedness of v in (1.2) as follows:

$$\beta_{\mathsf{Kn}} = \mathrm{diam}(\Omega) M_1^{-1} \left(\mathsf{Kn}(\|\sigma_a\|_{L^\infty} + \|\tilde{\sigma}_a\|_{L^\infty}) + \mathsf{Kn}^{-1}(\|\sigma_s\|_{L^\infty} + \|\tilde{\sigma}_s\|_{L^\infty}) \right). \tag{3.9}$$

Before going further, let us introduce some notation. We denote the set which consists the unit vectors in $\mathbb{R}^n$ by

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \ |x| = 1\}.$$

In X-ray tomography, a ray goes through the point $x \in \mathbb{R}^n$ and has the direction $\omega \in \mathbb{S}^{n-1}$. Integrating over this ray leads to the X-ray transform Xf of f, which is defined as

$$(Xf)(x, \omega) = \int_{\mathbb{R}} f(x + s\omega) ds \tag{3.10}$$

for every $\omega \in \mathbb{S}^{n-1}$. We denote a function g in the space $T\mathbb{S}^{n-1}$ by

$$\|g\|^2_{L^2(T\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} \|g(\cdot, \omega)\|^2_{L^2(T)} d\omega,$$

where $T = \{(x, \omega) \in \mathbb{R}^n \times \mathbb{S}^{n-1} : \ x \cdot \omega = 0\}$. We also denote the space

$$\partial\Omega \times \mathbb{S}^{n-1}_- = \{(x, \omega) \in \partial\Omega \times \mathbb{S}^{n-1} : \ n_x \cdot \omega < 0\}.$$

In particular, one can deduce that there is a constant $C_0 > 0$ such that

$$\|Xf\|_{L^2(T\mathbb{S}^{n-1})} \le C_0 \|Xf\|_{L^\infty(\partial\Omega\times\mathbb{S}^{n-1}_-)} \tag{3.11}$$

for all functions Xf in $L^\infty(\partial\Omega\times\mathbb{S}^{n-1}_-)$. Furthermore, from Theorem 3.1 in [31], for any function $f \in H^{-1/2}(\Omega)$ with compact support in Ω, there exists a constant $C_1 > 0$ such that

$$\|f\|_{H^{-1/2}(\Omega)} \le C_1 \|Xf\|_{L^2(T\mathbb{S}^{n-1})}. \tag{3.12}$$

Combining (3.11) and (3.12), this leads to

$$\|f\|_{H^{-1/2}(\Omega)} \le C_0 C_1 \|Xf\|_{L^\infty(\partial\Omega\times\mathbb{S}^{n-1}_-)}. \tag{3.13}$$

We use this estimate to show the following stability estimate.

Proposition 3.3. *We denote $\hat{v}'_0 = v'_0/|v'_0|$ to be the unit vector. Then*

$$\begin{aligned}\|\sigma_{\mathsf{Kn}} - \tilde\sigma_{\mathsf{Kn}}\|_{H^{-1/2}(\Omega)} &\le C_0 C_1 \|X(\sigma_{\mathsf{Kn}} - \tilde\sigma_{\mathsf{Kn}})\|_{L^\infty(\partial\Omega\times\mathbb{S}^{n-1}_-)} \\ &\le C_0 C_1 M_2 e^{\beta_{\mathsf{Kn}}} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*,\end{aligned}$$

where $\beta_{\mathsf{Kn}} > 0$ is defined in (3.9) *and M_2 is the upper bound of v stated in* (1.2).

Proof. For any $(x'_0, v'_0) \in \Gamma_-$, by applying the change of variable $t \mapsto |v'_0| t$ in (3.8), we obtain that

$$\|\mathcal{A} - \tilde{\mathcal{A}}\|_* \ge e^{-\beta_{\mathsf{Kn}}} |v'_0|^{-1} |X(\sigma_{\mathsf{Kn}} - \tilde\sigma_{\mathsf{Kn}})(x'_0, \hat{v}'_0)|.$$

The desired estimates follow by applying (3.13) to the above inequality. ∎

3.2. Proof of Theorem 1.2. The norm used in Proposition 3.3 is rather weak. Since we assume a priori that the medium is in the function space $\mathcal{P}$ with higher regularity defined in (1.10), the interpolation formula could be used to lift the stability estimate to a stronger result. Recall the interpolation formula which states the existence of constant C_2 so that

$$\|u\|_{H^s(\mathbb{R}^n)} \le C_2 \|u\|^{\theta}_{H^{s_1}(\mathbb{R}^n)} \|u\|^{1-\theta}_{H^{s_2}(\mathbb{R}^n)} \tag{3.14}$$

for any $s_1 < s_2$ and $s = \theta s_1 + (1-\theta)s_2$ with $0 < \theta < 1$. This constant purely depends on $C_2 = C_2(n, s_1, s_2)$. One simply needs to choose a special set of (s_1, s_2) to achieve the results in Theorem 1.2.

Proof of Theorem 1.2. For a fixed $r' > 0$, let $0 < r < r'$ and $s = \frac{3}{2} + r$. We set $s_1 = -\frac{1}{2}$ and $s_2 = \frac{3}{2} + r'$; then it is clear that $s_1 < s < s_2$ and there is a constant θ such that $s = \theta s_1 + (1-\theta)s_2$. Using the interpolation formula (3.14), we have

$$\|\sigma_{\mathsf{Kn}} - \tilde\sigma_{\mathsf{Kn}}\|_{H^{3/2+r}} \le C_2 \|\sigma_{\mathsf{Kn}} - \tilde\sigma_{\mathsf{Kn}}\|^{1-\theta}_{H^{3/2+r'}} \|\sigma_{\mathsf{Kn}} - \tilde\sigma_{\mathsf{Kn}}\|^{\theta}_{H^{-1/2}}.$$

From the hypothesis of Theorem 1.2, the parameters σ_a, σ_s, $\tilde\sigma_a$, and $\tilde\sigma_s$ are in $\mathcal{P}$ defined in (1.10). Combining with Proposition 3.3, the inequality could be further bounded by

$$\begin{aligned}\|\sigma_{\mathsf{Kn}} - \tilde\sigma_{\mathsf{Kn}}\|^{1/\theta}_{H^{3/2+r}} &\le \left(2C_2 M_3^{1-\theta} \left(\mathsf{Kn} + \mathsf{Kn}^{-1}\right)^{1-\theta}\right)^{1/\theta} \|\sigma_{\mathsf{Kn}} - \tilde\sigma_{\mathsf{Kn}}\|_{H^{-1/2}} \\ &\le \left(2C_2 M_3^{1-\theta} \left(\mathsf{Kn} + \mathsf{Kn}^{-1}\right)^{1-\theta}\right)^{1/\theta} C_0 C_1 M_2 e^{\beta_{\mathsf{Kn}}} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*.\end{aligned}$$

According to the definition of β_{Kn}, one has

$$\beta_{\mathsf{Kn}} \leq C(\mathsf{Kn} + \mathsf{Kn}^{-1})$$

for some constant C independent of Kn. Thus, by applying Sobolev embedding theorem, we have the following L^∞ estimate for $\sigma_{\mathsf{Kn}} - \tilde{\sigma}_{\mathsf{Kn}}$:

$$\|\sigma_{\mathsf{Kn}} - \tilde{\sigma}_{\mathsf{Kn}}\|_{L^\infty}^{1/\theta} \leq \left(2C_2 M_3^{1-\theta} \left(\mathsf{Kn} + \mathsf{Kn}^{-1}\right)^{1-\theta}\right)^{1/\theta} C_0 C_1 C_3 M_2 e^{C(\mathsf{Kn}+\mathsf{Kn}^{-1})} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*, \tag{3.15}$$

where the constants C_j, $j = 0, \ldots, 3$, are independent of Kn.

Finally, we are ready to show (1.11) and (1.12). From the definition of σ_{Kn} and (3.15), we derive

$$\mathsf{Kn}\|\sigma_a - \tilde{\sigma}_a\|_{L^\infty} \leq \mathsf{Kn}^{-1}\|\sigma_s - \tilde{\sigma}_s\|_{L^\infty} + C\left(\mathsf{Kn} + \mathsf{Kn}^{-1}\right)^{1-\theta} e^{C\theta(\mathsf{Kn}+\mathsf{Kn}^{-1})} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\theta,$$

where C is independent of Kn. In particular, we obtain the following estimate for $\sigma_a - \tilde{\sigma}_a$:

$$\|\sigma_a - \tilde{\sigma}_a\|_{L^\infty} \leq \mathsf{Kn}^{-2}\|\sigma_s - \tilde{\sigma}_s\|_{L^\infty} + C\mathsf{Kn}^{-1}\left(\mathsf{Kn} + \mathsf{Kn}^{-1}\right)^{1-\theta} e^{C\theta(\mathsf{Kn}+\mathsf{Kn}^{-1})} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\theta. \tag{3.16}$$

On the other hand, one can also derive an estimate for $\sigma_s - \tilde{\sigma}_s$:

$$\|\sigma_s - \tilde{\sigma}_s\|_{L^\infty} \leq \mathsf{Kn}^2\|\sigma_a - \tilde{\sigma}_a\|_{L^\infty} + C\mathsf{Kn}\left(\mathsf{Kn} + \mathsf{Kn}^{-1}\right)^{1-\theta} e^{C\theta(\mathsf{Kn}+\mathsf{Kn}^{-1})} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\theta. \tag{3.17}$$

Assume that $\|\mathcal{A} - \tilde{\mathcal{A}}\|_* < 1$. Under the assumption that scattering dominates absorption in a region of interest, we consider the case

$$\mathsf{Kn} < \min\{|\log(\|\mathcal{A} - \tilde{\mathcal{A}}\|_*)|^{-\alpha},\ 1\} \tag{3.18}$$

for some constant $\alpha > 0$. Then from (3.17) and from the hypothesis that $\sigma_a,\ \tilde{\sigma}_a \in \mathcal{P}$, it leads to

$$\begin{aligned}
\|\sigma_s - \tilde{\sigma}_s\|_{L^\infty} &\leq C\mathsf{Kn}^2 M_3 + C\mathsf{Kn}\left(\mathsf{Kn} + \mathsf{Kn}^{-1}\right)^{1-\theta} e^{C\theta\mathsf{Kn}^{-1}} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\theta \\
&\leq C\mathsf{Kn}|\log(\|\mathcal{A} - \tilde{\mathcal{A}}\|_*)|^{-\alpha} + C\mathsf{Kn}\left(\mathsf{Kn} + \mathsf{Kn}^{-1}\right)^{1-\theta} e^{C\theta\mathsf{Kn}^{-1}} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\theta. \qquad (3.19)
\end{aligned}$$

On the other hand, one can also use a similar argument to derive the following estimate based on (3.16), that is,

$$\|\sigma_a - \tilde{\sigma}_a\|_{L^\infty} \leq C\mathsf{Kn}^{-3}|\log(\|\mathcal{A} - \tilde{\mathcal{A}}\|_*)|^{-\alpha} + C\mathsf{Kn}^{-1}\left(\mathsf{Kn} + \mathsf{Kn}^{-1}\right)^{1-\theta} e^{C\theta\mathsf{Kn}^{-1}} \|\mathcal{A} - \tilde{\mathcal{A}}\|_*^\theta. \tag{3.20}$$

This completes the proof of the theorem. ∎

We remark below some observations about the stability estimates which are derived in the proof of Theorem 1.2.

Remark 3.1. We briefly discuss the derived estimates (3.16) and (3.17), and the results obtained in [12]. Assume that $\sigma_s = \tilde{\sigma}_s$ is given. This is the setup in sections 3.1 and 3.2 in [12]. In this setting, we study the stability of the coefficient σ_a depending on the Knudsen number Kn. Then we obtain a similar result as in [12] where the linearized inverse problem is considered. In particular, we conclude from (3.16) that the difference of $\sigma_a - \tilde{\sigma}_a$ could become larger if Kn is decreasing. This also means that a smaller Kn leads to worse distinguishability of the absorption coefficient.

When $\sigma_a = \tilde{\sigma}_a$ is known, similar to the observation in section 4.3 in [12], we have from (3.17) that the difference of $\sigma_s - \tilde{\sigma}_s$ might also increase as Kn shrinks.

Remark 3.2. We specifically mention the goal of the estimates (3.19) and (3.20). In the zero limit of Kn, the RTE becomes the diffusion equation, whose DtN map is shown to reconstruct the media with logarithmic instability. This is reflected in the theorem above as well. When Kn is sufficiently small, the $\log(\|\mathcal{A} - \tilde{\mathcal{A}}\|_*)$ term appears in the estimate. Therefore, the stability estimate is connected to the log type ill-posedness which is seen in the inverse diffusion problem.

4. Numerics. We present numerical evidence in this section. We utilize a simpler model in 2D:

$$v \cdot \nabla f = \cos\theta \partial_x f + \sin\theta \partial_y f = \frac{\sigma_s}{\mathsf{Kn}} \left(\langle f \rangle - f \right), \tag{4.1}$$

where $\langle f \rangle_v = \int f \mathrm{d}\theta$ with $\mathrm{d}\theta$ is normalized. This is the critical case in the sense that the effective absorption is set to be zero. To demonstrate stability, we consider two sets of media with the absorption coefficients:

$$\sigma_s = 1 \quad \text{for } (x, y) \in \Omega\,, \quad \tilde{\sigma}_s = \begin{cases} 1 & \text{for } (x, y) \in B, \\ 1 + z & \text{for } (x, y) \in \Omega \backslash B, \end{cases}$$

where $\Omega = [0, 0.6]^2$ and B is a ball centered at $(0.3\,, 0.3)$ with radius 0.2. Clearly, $\|\sigma_s - \tilde{\sigma}_s\|_{L^\infty} = z$. In computation we choose z to be $0.1 \times \{1, 1/2, 1/4, 1/8\}$.

We denote the associated albedo operator by $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively. They map the incoming data to the outgoing data. We also denote $\mathcal{A}_1$ and $\tilde{\mathcal{A}}_1$ to be the leading order expansion of the albedo operator, as defined in (2.4). We will show numerical evidence from the following three aspects:

1. $\|\mathcal{A}_1\|_*$ decays exponentially fast with respect to Kn;
2. For a fixed Kn, $\|\mathcal{A} - \tilde{\mathcal{A}}\|_*$ grows in a Lipschitz manner with respect to $z = \|\sigma_s - \tilde{\sigma}_s\|_{L^\infty}$;
3. For a fixed z, $\|\mathcal{A} - \tilde{\mathcal{A}}\|_*$ blows up exponentially fast with respect to Kn.

Through the entire computation, we use $\mathrm{d}x = 0.025$ and $\mathrm{d}\theta = 2\pi/24$ to resolve all possible small scales. To obtain the operator norm, we need to numerically exhaust all possible boundary conditions. We then term the collection of discrete boundary conditions $\mathcal{S}$. In addition, the numerical operator norm is defined as follows:

$$\|\mathcal{A}_1\|_* = \sup_{\phi \in \mathcal{S}} \frac{\|\mathcal{A}_1 \phi\|_{L^1(\Gamma_+)}}{\|\phi\|_{L^1(\Gamma_-)}} \quad \text{and} \quad \|\mathcal{A} - \mathcal{A}_1\|_* = \sup_{\phi \in \mathcal{S}} \frac{\|\mathcal{A}\phi - \mathcal{A}_1 \phi\|_{L^1(\Gamma_+)}}{\|\phi\|_{L^1(\Gamma_-)}}\,. \tag{4.2}$$

To obtain $\mathcal{A}_1 \phi$ in (4.2), we numerically solve

$$v \cdot \nabla_x f = -\frac{1}{\mathsf{Kn}} \sigma_s f\,, \quad \text{with} \quad f|_{\Gamma_-} = \phi\,, \tag{4.3}$$

and then confine the solution on Γ_+:

$$\mathcal{A}_1 \phi = f|_{\Gamma+}\,. \tag{4.4}$$

Similarly one can obtain $\mathcal{A}\phi$ by replacing (4.3) with (4.1).

Exponential decay in Kn of $\mathcal{A}_1$. In the first experiment, we set $z = 0.1$ and choose $\mathsf{Kn} = 2^k$ with k varying from 1 to -3 in the RTE. We also evaluate $\|\mathcal{A}_1\|_*$ by using the operator norm (4.2). Numerically, we observe that the operator norm of $\mathcal{A}_1$ decays with respect to $1/\mathsf{Kn}$, which is seen in Figure 1. In particular, the computation suggests that

$$\|\mathcal{A}_1\|_* \sim e^{-\frac{0.1}{\mathsf{Kn}}}\,.$$

In the zero limit of Kn, the operator norm is extremely small.

We also numerically evaluate the operator norm of $\mathcal{A} - \mathcal{A}_1$ and study its dependence on Kn. The numerical evidence shows that as Kn shrinks to zero, the discrepancy between $\mathcal{A}$ and $\mathcal{A}_1$ grows, which agrees with the observation made in [4, 35]. We emphasize that $\mathcal{A}_1$ contains the most singular information in $\mathcal{A}$. By separating $\mathcal{A}_1$ from $\mathcal{A}$, one is able to recover the absorption coefficients (σ_s here). In the zero limit of Kn, $\mathcal{A}$ and $\mathcal{A}_1$ have large discrepancy, meaning $\mathcal{A}_1$ has very limited contribution in $\mathcal{A}$. This could potentially make the separation harder, which leads to the worse reconstruction.

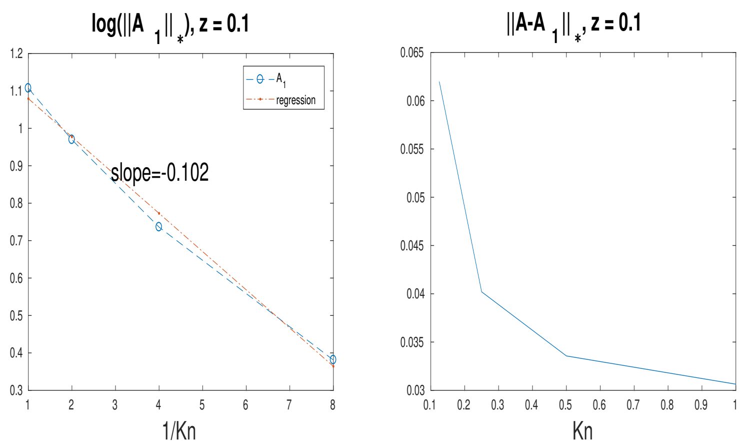

FIG. 1. *The plot on the left shows that* $\ln \|\mathcal{A}_1\|_*$ *linearly decays as* $1/\mathsf{Kn}$ *increases. The plot on the right shows that* $\|\mathcal{A} - \mathcal{A}_1\|_*$ *blows up as* Kn *converges to zero.*

Lipschitz continuity in z. In the second experiment we set $\mathsf{Kn} = 1$ and study the dependence of $\|\mathcal{A} - \tilde{\mathcal{A}}\|_*$ on $z = \|\sigma_s - \tilde{\sigma}_s\|_{L^\infty}$. The numerical experiment suggests that the discrepancy between the two albedo operators increases linearly with respect to z, which agrees with our Hölder continuity result; see Figure 2.

Exponential blow-up in Kn. In the third experiment, we fix $z = 0.025$ and compare the difference between the two albedo operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ as a function of Kn. It is expected that the difference between the two albedo operators behaviors is $e^{-\frac{c}{\mathsf{Kn}}}$, according to Theorem 1.2, which is also what we observe numerically. As seen in Figure 3, $\ln\left(\|\mathcal{A} - \tilde{\mathcal{A}}\|_*\right)$ is a linear function of $1/\mathsf{Kn}$, with slope -0.05. This indicates that

$$\ln \|\mathcal{A} - \tilde{\mathcal{A}}\|_* \sim -\frac{0.05}{\mathsf{Kn}} \quad \Rightarrow \quad \|\mathcal{A} - \tilde{\mathcal{A}}\|_* \sim e^{-\frac{0.05}{\mathsf{Kn}}}\,.$$

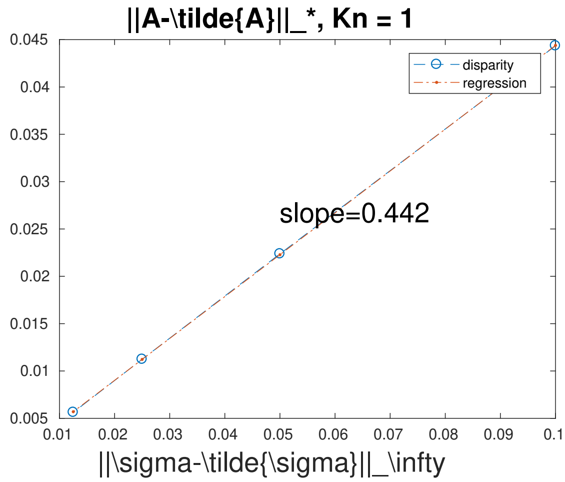

FIG. 2. *The plot shows that for a fixed* $\mathsf{Kn} = 1$*, larger* $\|\sigma_s - \tilde{\sigma}_s\|_{L^\infty}$ *leads to larger* $\|\mathcal{A} - \tilde{\mathcal{A}}\|_*$*. In particular, they form a linear dependence.*

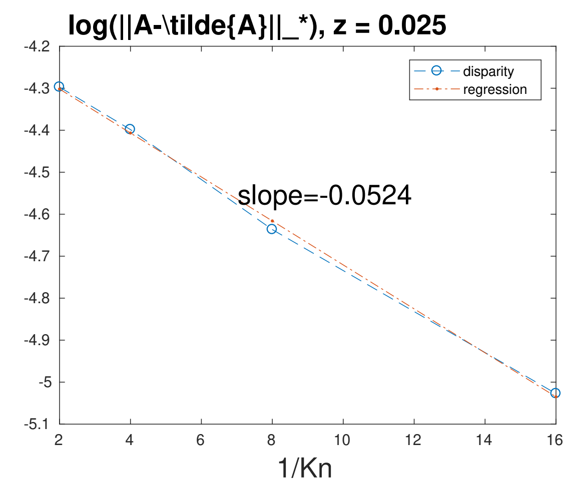

FIG. 3. *The plot shows that* $\|\mathcal{A} - \tilde{\mathcal{A}}\|_* \sim e^{-\frac{0.05}{\mathsf{Kn}}}$ *when* $\mathsf{Kn} \to 0$.

Before finishing the section, we emphasize that the numerical experiment can be done for only limited choices of Kn, σ_s, and z. However, the theory gives an upper bound for all possible combinations. It is also possible to design a special medium whose inverse stability is better than the one suggested by the theorem.

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